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ON CONTROL PROBLEMS WITH RESTRICTED COORDINATES

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The approach developed in monograph [1] is used to consider the problem of control for a linear system with bounded phase coordinates. The properties of the solutions and of the boundary conditions for the corresponding associated system are discussed. Additional information is obtained about the Lagrange coefficients; this information can be used to reduce solutions of the initial multidimensional problem to the minimization of a function of a finite number of variables.

1. Formulation of the problem. Let us consider the controlled motion described by the equation

$$dx/dt = A(t)\,x + B(t)\,u + w(t) \tag{1.1}$$

Here the vector x is n-dimensional, the control u is r-dimensional, and the matrices $A(t)$, $B(t)$ and the perturbation n-vector $w(t)$ are continuous.

Problem 1.1. We are given system (1.1), boundary conditions $x(t_\alpha) = x_\alpha$, $x(t_\beta) = x_\beta$, and the restrictions

$$\operatorname{vrai\,max}_t |u_j(t)| \leqslant \nu_j, \quad t_\alpha \leqslant t \leqslant t_\beta \quad (j = 1, \ldots, r) \tag{1.2}$$

on the control $u \in U$, and

$$|x_k(t)| \leqslant f_k(t) \quad (k = 1, \ldots, m) \tag{1.3}$$

on the coordinates $x(t) \in X(t)$.

The functions $f_k(t)$ are absolutely continuous and positive. We are to bring system (1.1) from x_α to x_β in the minimum time $t_\beta^0 - t_\alpha$ under restrictions (1.2), (1.3).

2. The solvability conditions. The maximum principle. The result of the present section is valid for all closed convex restrictions $u \in U$, $x \in X$ on the instantaneous values of the controls and of the first m coordinates, provided that zero is an interior point of both U and X.

Let us assume that t_β is fixed and consider the following moment calculation problem:

$$\begin{aligned} &\int h_j(t_\beta, \tau)\, u(\tau)\, d\tau = c_{j\beta} && (j = 1, \ldots, n) \\ &\int h_k(t_i, \tau)\, u(\tau)\, d\tau + z_k^{(i)} = c_{ki} && (k = 1, \ldots, m) \end{aligned} \tag{2.1}$$

for

$$u \in U, \quad z^{(i)} \in Z(t_i) \qquad (Z = -X) \tag{2.2}$$

Here $\{t_i\}$, the set of points (e. g. of the form $t_\alpha + i(N)(t_\beta - t_\alpha)\, 2^{-N}$, where $i(N) \leqslant 2^N$, $N = 1, 2, 3, \ldots$), is dense everywhere in the segment (*) $[t_\alpha, t_\beta]$;

*) See Note at the bottom of next page.

the functions $h_j(t, \tau)$ are the jth rows of an ($n \times r$)-matrix

$$h(t, \tau) = S(\tau, t) B(\tau)$$

$$(dS(\tau, t) / dt = -S(\tau, t) A(\tau), \qquad S(t, t) = E)$$

where $h_j(t, \tau) \equiv 0$ if $\tau \geqslant t$; the numbers c_{ki} are the components of the vector

$$c(t_i) = -S(t_\alpha, t_i) x_\alpha - \int_{t_\alpha}^{i} S(\tau, t_i) w(\tau) d\tau$$

$$c_{k\beta} = c_k(t_\beta), \quad \left(\sum_k c_{k\beta}^2 \neq 0\right), \qquad z^{(i)} = -x(t_i)$$

Moment problem (2.1)–(2.3) is related in a certain way [1] to initial Problem 1.1. Specifically, the time-optimal control for Problem 1.1 is the solution of (2.1), (2.2) for the smallest of the t_β for which the problem is solvable.

The solvability conditions for problem (2.1), (2.2) can be obtained by the methods of functional analysis. In fact, let us set

$$\gamma[h] = \max_u \; hu (u \in U, h \in R_n) \tag{2.3}$$

$$\gamma_1[l(t_i)] = \max_z \; l(t_i) z \; (z \in Z(t_i)) \tag{2.4}$$

This enables us to consider problem (2.1)–(2.3) as the problem of constructing the linear operation (u, z) on the elements $h_j(t_\beta, \tau)$, $h_k(t_i, \tau)$ of the r-vector space L_2 and on the elements $e^{(i)} = \{e_j^{(i)}, j = 1, \ldots, N, \ldots\}$ ($e_j^{(i)} = (0, \ldots 0)$ if $i \neq j$; $e_j^{(i)} = (0, \ldots, E, 0, \ldots, 0)$ if $i = j$) of the m-vector space l_2. By virtue of the closure and convexity of the set of functions $u(t)$ from L_2 restricted by the condition $u \in U$ and of the vectors $\{z^{(i)}\}$ from l_2 satisfying (2.2), restrictions (2.2) can be replaced by the inequalities [1, 2]

$$\int h(t) u(t) dt \leqslant \int \gamma[h(t)] dt = p[h] \tag{2.5}$$

$$\sum_{i=1}^{\infty} l(t_i) z^{(i)} \leqslant \sum_{i=1}^{\infty} \gamma_1[l(t_i)] = p_1[l] \tag{2.6}$$

These inequalities must be satisfied for all h from L_2 and all $l^{(i)}$ from l_2. The required operation (u, z) is therefore majorated by the sublinear functional $(p[h], p_1[l])$. This makes it possible to use the Hahn-Banach theorem [2], which means that the necessary and sufficient condition of solvability of the problem of moments can be reduced by means of the familiar procedure [3] to ensuring fulfilment of the inequality

$$\begin{aligned} &\int \gamma\left[\sum_{j=1}^{n} l_j h_j(t_\beta, \tau) + \sum_{\Omega_{N,k}} l_k^{(i)} h_k(t_i, \tau)\right] d\tau + \\ &+ \sum_{\Omega_N} \gamma_1[l^{(i)}] - \sum_{j=1}^{n} l_j c_{\beta j} - \sum_{\Omega_{N,k}} l_k^{(i)} c_{ki} \geqslant 0 \end{aligned} \tag{2.7}$$

for all finite sets Ω_N of the vectors $l^{(i)}$ and all n-vectors $\{l_j\}$.

The Stieltjes integral [4] and the bounded vector functions $\Lambda = \{\Lambda_k(t)\}$ continuous from the right enable us to rewrite (2.7) as the equivalent inequality

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*) The omission of integration limits here and throughout the discussion to follow means that the lower limit is the left-hand end t_α and the upper limit the right-hand end t_β of the segment $[t_\alpha, t_\beta]$. The limits in other cases are specified.

$$\int \gamma \Big[\sum_{j=1}^{n} l_j h_j (t_\beta, \tau) + \sum_{k=1}^{m} \int_{\tau}^{t_\beta} h_k (t, \tau)\, d\Lambda_k (t) \Big] d\tau +$$
$$+ \int \gamma_1 [d\Lambda] - \sum_{j=1}^{n} l_j c_j (t_\beta) - \int \sum_{k=1}^{m} c_k (t)\, d\Lambda_k (t) \geqslant 0 \tag{2.8}$$

Problem (2.1),(2.3) is solvable if and only if inequality (2.8) is valid for any vector l and for any bounded m-vector function $\Lambda (t)$. Here

$$\int \gamma_1 [d\Lambda] = \sup \sum_{i=1}^{M-1} \gamma_1 [\Lambda (t_{i+1}) - \Lambda (t_i)]$$

over all possible finite decompositions $t_\alpha = t_1 < \ldots < t_M = t_\beta$ of the segment $[t_\alpha, t_\beta]$. The sufficiency of condition (2.8) clearly follows from (2.8). The necessity of condition (2.8) can be established indirectly with allowance for the possibility of approximating any bounded function $\Lambda (t)$ by means of a suitably chosen piecewise-constant function. The integrals

$$\int \gamma [\ \ ]\, d\tau, \qquad \int \gamma_1 [d\Lambda]$$

are necessarily nonnegative, since both U and X contain zero as an interior point. But (2.8) then implies the equivalent condition

$$\nu = \inf_{l,\Lambda} \Big\{ \int \gamma \Big[\sum_{j=1}^{n} l_j h_j (t_\beta, \tau) +$$
$$+ \int_{\tau}^{t_\beta} \sum_{k=1}^{m} h_k (t, \tau)\, d\Lambda_k (t) \Big] d\tau + \int \gamma_1 [d\Lambda] \Big\} = \inf_{l,\Lambda} \Psi (l, \Lambda) \geqslant 1 \tag{2.9}$$

for

$$\sum_{j=1}^{n} l_j c_{\beta j} + \int \sum_{k=1}^{m} c_k (t)\, d\Lambda_k (t) = 1 \tag{2.10}$$

Let us note immediately that the inf in (2.10) is necessarily attained at a nonzero element $\eta = (l^\circ, \Lambda^\circ (t))$. In fact, let us consider the minimizing sequence $l^{(N)}, \Lambda^{(N)}(t)$. Setting $c (t_\beta) \neq 0$, we see that ν is a finite quantity. Setting $\eta^{(N)} = (l^{(N)}, \Lambda^{(N)})$ in the left side of (2.9), we obtain the numbers ν_N, where $\nu_N \to \nu$. This implies directly that the functions $\Lambda_k^{(N)}(t)$ are uniformly bounded in norm: var $\Lambda_k^{(N)} (t) \leqslant k_1$ for all $N \geqslant N_0$, $k = 1, \ldots, m$, by virtue of the boundedness of the quantity

$$\int \gamma_1 [d\Lambda^{(N)}]$$

Now let us assume that the functions $h_j (t_\beta, \tau)$ are linearly independent. Such an assumption is quite legitimate in the problem of control over all the coordinates. It is equivalent to the controllability condition [1] for system (1.1). Under the above assumption

$$\int \gamma \Big[\sum_{j=1}^{n} l_j h_j (t_\beta, \tau) \Big] d\tau \to \infty$$

if the Euclidean norm $\| l \| \to \infty$. The convergence property $\nu_N \to \nu$ and the boundedness of the functions $\Lambda^{(N)}$ now implies that the set $l^{(N)}$ is bounded: $\| l^{(N)} \| \leqslant k_2$ if $N \geqslant N_0$. By virtue of the compactness of a sphere in the finite-dimensional space R_n and the weak compactness of a sphere in the space of bounded functions (see the Helly theorems in [4]), we can (retaining our old notation) isolate a subsequence $\eta^{(N)} = (l^{(N)}, \Lambda^{(N)} (t))$, which converges ($N \to \infty$) to the element $\eta^\circ = (l^\circ, \Lambda^\circ (t))$ in such a way that $l^{(N)} \to l^\circ$ in norm and $\Lambda^{(N)} \to \Lambda^\circ$ in the sense of weak convergence (or convergence "in the large"). Condition (2.10) remains valid for the limiting element, i.e.

$$\sum_{j=1}^{n} l_j^{\circ} c_j(t_\beta) + \int \sum_{k=1}^{m} c_k(t)\, d\Lambda_k^{\circ}(t) = 1 \tag{2.11}$$

The integrands in (2.9) indicate that $\nu = \Psi(l^\circ, \Lambda^\circ(t))$ (our reasoning here is the same as that of [5]); specifically, we have the relation

$$\int \gamma_1 [d\Lambda^{(N)}] \to \int \gamma_1 [d\Lambda^\circ]$$

With allowance for (2.11) we infer from this that the minimizing element of (2.9) is a nonzero element. This property remains valid even if the functions $h_j(t_\beta, \tau)$ are linearly dependent but the vector $c(t_\beta)$ is such that the initial problem is solvable in the absence of restrictions on the coordinates (e.g. see [1], Sect. 15).

Let Problem 1.1 be solvable for some t_β and $\nu[t_\alpha, t_\beta] \geqslant 1$. For $t_\beta = t_\alpha$ we have $\nu[t_\alpha, t_\alpha] = 0$.

Since $\nu[t_\alpha, t_\beta]$ is continuous in t_β, there exists a smallest number t_β° such that $\nu[t_\alpha, t_\beta^\circ] = 1$. This number yields the solution of Problem 1.1.

We can rewrite problem (2.9), (2.10) in a different notation. Setting (see Note 2.1)

$$ds(t) = -s(t) A(t)\, dt + d\Lambda(t) \tag{2.12}$$

$$s_\beta = s(t_\beta), \qquad \Lambda_i(t) = \text{const}, \qquad \text{if} \quad i > m, \quad s(t_\alpha) = s_\alpha$$

we obtain instead of (2.9) the condition

$$\min_{s(t)} \left\{ \int \gamma [s(t) B(t)]\, dt + \int \gamma_1 [d\Lambda] \right\} = 1 \tag{2.13}$$

where the minimum is taken over all the motions of associated system (2.12) restricted by the equation

$$s_\beta x_\beta - s_\alpha x_\alpha + \int s(t) w(t)\, dt = 1 \tag{2.14}$$

The minimum motion $s^\circ(t)$ which yields the extremum of (2.13) under condition (2.14) is, of course, that which is determined by the vector $s_\beta^\circ = l^\circ$ and by the function $\Lambda^\circ(t)$ forming the extremal element (2.9), (2.10).

Making use of (2.3), (2.4), we infer directly that equality in (2.13) applies when the control $u^\circ(t)$ satisfies the maximum principle

$$\int s^\circ(t) B(t) u^\circ(t)\, dt = \max_u \int s^\circ(t) B(t) u(t)\, dt \qquad (u \in U) \tag{2.15}$$

or

$$s^\circ(t) B(t) u^\circ(t) = \max_u s^\circ(t) B(t) u(t) \qquad (u \in U) \tag{2.16}$$

and when the trajectory $x^\circ(t)$ satisfies the maximum condition

$$-\int x^\circ(t)\, d\Lambda^\circ(t) = \max_x \int x(t)\, d\Lambda^\circ(t) \qquad (x(t) \in Z(t)) \tag{2.17}$$

The following theorem summarizes the above discussion.

Theorem 2.1. Problem 1.1 is solvable if and only if condition (2.13), where γ, γ_1 are defined by (2.3), (2.4), is satisfied on the motions $s(t)$ of associated system (2.12) under restriction (2.14). The optimal control $u^\circ(t)$ satisfies maximum principle (2.15) (or (2.16)) on the minimum motion of problem (2.13), (2.14). The optimal trajectory satisfies maximum condition (2.17) on the function $\Lambda^\circ(t)$ which generates the minimum motion $s^\circ(t)$ as well as the boundary condition s_β°.

The maximum principle has meaning in this problem provided that $h^\circ = s^\circ(t) B(t) \not\equiv 0$

on $[t_\alpha, t_\beta]$. This condition is discussed in Sect. 3 below.

If the function $\Lambda^\circ(t)$ is absolutely continuous, then we can speak of the "regular case" of Problem 1.1; if not, we shall call the problem "irregular".

Note 2.1. The expression $d\Lambda^\circ = \lambda^\circ(t)\,dt$ is valid in the regular case of Problem 1.1. Here we can speak of the "ordinary derivative" $d\Lambda^\circ / dt = \lambda^\circ(t)$ of the function $\Lambda^\circ(t)$. The quantity $s^\circ(t)$ is continuous in this case. Relation (2.12) is the ordinary differential equation

$$ds / dt = -sA(t) + \lambda(t) \tag{2.18}$$

In the irregular case $d\Lambda / dt$ can be interpreted only as a generalized derivative, and (2.18) as an equality of distributions [3, 6], i.e. as some generalized equation one of whose solutions may be the already discontinuous function $s(t)$. The above remarks are also basic to the interpretation of Eq. (2.12).

Note 2.2. The definition of the quantity $\gamma_1(t)$ implies, by virtue of (2.17), that if $\Lambda^\circ(t)$ has a discontinuity $\Lambda^\circ(t'+0) - \Lambda^\circ(t'-0) = \lambda$ at $t = t'$, then the m-dimensional vector λ is supporting to the convex set $X(t')$ at the point $x^\circ(t')$ lying on the boundary of the set $X(t')$.

Note 2.3. Under specific restrictions (1.2), (1.3) we have

$$\gamma[h] = \sum_{j=1}^{r} \nu_j |h_j|, \qquad \gamma_1[l(t_i)] = \sum_{k=1}^{m} f_k(t_i)\,|l_k|$$

Note 2.4. In solving problem (2.13), (2.14), we can replace unity in the right sides of the corresponding conditions by some constant $c > 0$ whose choice can be based on such considerations as convenience of calculation.

Note. 2.5. Suitably modified, the above procedure remains valid for the problem of bringing system (1.1) from one convex manifold to another in the minimum time in the case of piecewise-absolutely-continuous functions $f_k(t)$.

3. Properties of the solutions. Let $X'(t)$ be the set of boundary points for $X(t)$.

Lemma 3.1. If $x^\circ(t) \in X(t) - X'(t)$ for $t \in e$, then

$$\Lambda^\circ(t) \equiv \text{const}$$

for $t \in e$.

Under the conditions of the lemma we have

$$-\int_e x^\circ(t)\,d\Lambda^\circ(t) < \max_{z(t)\in Z(t)} \int_e x(t)\,d\Lambda^\circ(t) = \int_e \gamma_1[d\Lambda^\circ]$$

which together with (2.17) yields the condition $\Lambda^\circ(t) = \text{const}$ for $t \in e$. Proceeding with our argument, we narrow somewhat the class of functions $f_i(t)$ defining the set $X(t)$.

Hypothesis 3.1. The functions $f_i(t)$ are such that each of the equations

$$\sum_{j=1}^{n} s_{ij}(t_\delta, t_\gamma)\,x_j(t_\gamma) \pm \int_{t_\gamma}^{t_\delta} \sum_{j=1}^{r} h_{ij}(t_\delta, \xi)\,\nu_j\,d\xi = f_i(t_\delta)$$
$$(i = 1, \ldots, m) \tag{3.1}$$
$$t_\alpha \leqslant t_\gamma \leqslant t_\delta, \qquad x_s(t_\gamma) = f_s(t_\gamma) \quad (s = 1, \ldots, m)$$

can be fulfilled for each t_γ, $x_j(t_\gamma)$ only on a set of values $\{t_\delta\}$ of zero measure.

Roughly speaking, this means that no piece of the trajectory $x(t)$ of system (1.1) constructed for $u_j = \nu_j$ (or $-\nu_j$), can lie entirely on $X'(t)$.

Let us cite briefly certain sufficient conditions under which Hypothesis 3.1 holds.

For example, let Hypothesis 3.1 not be fulfilled for the i'th coordinate and let the function $f_i(t)$ be piecewise-linear. Then it is necessarily the case that in some interval

$$dx_i / dt = a^{(i)}x + b^{(i)} \quad u = \alpha = \text{const}$$

where $u_j = \nu_j$ or $u_j = -\nu_j$, and where the symbol $a^{(i)}$ represents the ith row of the matrix A. Let Q_i be the matrix consisting of the ith rows of the matrices $A, A^2, \ldots, \ldots, A^n$. Let us assume that $a^{(i)} \neq 0$ (so that $Q_i \neq 0$). Differentiating dx_i / dt, we obtain

$$Q_i x = p, \qquad Q_i A x = r \tag{3.2}$$

where the n-vectors p, r are given by the formulas

$$p_1 = \alpha - b^{(i)}u, \quad p_s = -a^{(i)} A^{s-2} bu \quad (s = 2, \ldots, n)$$

$$r_k = -a^{(i)}A^{k-1}bu \qquad (k = 1, \ldots, n)$$

Let the n-vectors $e^{(1)}, \ldots, e^{(j)}$ $(e^{(i)} = q^{(i)} A, j \leqslant n)$ be the basis of the subspace stretched over the vectors $q^{(i)}A$ [7]. Hypothesis 3.1 is fulfilled if systems (3.2) do not have a common solution. By virtue of the properties of the matrices Q_i, $Q_i A$, this is the case if in at least one case $p^j \neq d^j$, where p^j, d^j are the coefficients of the expansions of the vectors p, r in the basis vectors $e^{(j)}$.

Let $a^{(i)} = 0$. Hypothesis 3.1 is fulfilled if $b^{(i)} u \neq \alpha$ for $u_j = \nu_j$ or $u_j = -\nu_j$. This yields the following statement.

Lemma 3.2. Let the functions $f_i(t)$ be piecewise-linear, and let the matrices A, B be constant. Hypothesis 3.1 is then fulfilled for $Q_i \neq 0$ if at least one of the inequalities $p^{(j)} \neq d^{(j)}$ is fulfilled for the coefficients $p^{(j)}$, $d^{(j)}$ of the expansion of the vectors p, r in the basis $e^{(j)}$. Otherwise (if $Q_i = 0$) the sufficient condition is $b^{(i)} u \neq \alpha$.

We note that the condition of the first part of the lemma is especially simple if the matrix Q_i is nonsingular.

Lemma 3.3. If Hypothesis 3.1 is valid for the system (1.1) and if at least one of the inequalities $|x_k^\circ(t)| = f_k(t)$, $\tau_1 \leqslant t \leqslant \tau_2$, is fulfilled, then the condition $h_j^\circ \equiv 0$ for $\tau_1 \leqslant t < \tau_2$ is valid for at least one of the components h_j of the function $h^\circ(t) = s^\circ(t) B$.

The statement of the lemma follows from the maximum principle (2.16) with allowance for restrictions (1.2) imposed on the control.

From now on we assume that system (1.1) is completely controllable in the reinforced sense [1]. This means that for every fixed i the system $\{h_{ij}(t_\beta, \tau)\}$ of the ith components of the functions $h_j(t_\beta, \tau)$ $j = 1, \ldots, n$, is linearly independent; each combination

$$\sum_{j=1}^{n} l_j h_{ij}(t_\beta, \tau) \qquad \left(\sum_{j=1}^{n} l_i^2 \neq 0\right)$$

can equal zero only on a set of zero measure. In the stationary case this condition is equivalent to the linear independence of the system of vectors $\{b^{(i)}, \ldots, A^{n-1}b^{(i)}\}$ for every $i = 1, \ldots, r$.

Lemma 3.4. If A, B are constant, if system (1.1) is completely controllable in the reinforced sense, and if at least one of the functions $h_j^\circ(t)$ vanishes identically in the interval $\tau_1 < t < \tau_2$, then at least one of the conditions $|x_i^\circ(t)| = f_i(t)$ is fulfilled in this interval.

First let us show that $s_\beta^\circ \neq 0$. Setting $x_\alpha = 0$ (so that $x_\beta \neq 0$), we find from (2.14) that $s_\beta^\circ \neq 0$. Let $x_\alpha \neq 0$ but $s_\beta^\circ = 0$. Writing out conditions (2.13), (2.14), we find that

$$\min\left\{\int_{t_\alpha}^{t_\beta^\circ} \gamma\left[s_\alpha S(t, t_\alpha) + \int_{t_\alpha}^{t_\beta^\circ} d\Lambda(\xi) S(t,\xi) B\right] + \int_{t_\alpha}^{t_\beta^\circ} \gamma_1[d\Lambda]\right\} \geqslant 1 \qquad \{s(t)\} \tag{3.3}$$

for

$$s_\beta x_\beta - s_\alpha x_\alpha + \int_{t_\alpha}^{t_\beta^\circ} s(t)\, w(t)\, dt = 1$$

If the trajectory $x^\circ(t)$ does not emerge onto the restriction, then $\Lambda \equiv \text{const}$ and $s_\beta = s_\alpha S(t_\beta, t_\alpha) \neq 0$. Let $t' \leqslant t_\beta$ be the instant of the first emergence onto the restriction in motion from t_β to t_α . Then $\Lambda^\circ(t) \neq \text{const}$ when $t' - \varepsilon \leqslant t \leqslant t' + \varepsilon$ for any $\varepsilon > 0$, and instead of (3.3) we have

$$\min\left\{\int_{t_\alpha}^{t'} \gamma[s(t) B]\, dt + \int_{t_\alpha}^{t'} \gamma_1[d\Lambda]\right\} \geqslant 1 \qquad \{s(t)\} \tag{3.4}$$

for

$$\int_{t_\alpha}^{t_\beta} s(t)\, w(t)\, dt - s_\alpha x_\alpha = 1$$

We can interpret relation (3.4) as the condition of solvability of the problem of bringing system (1.1) from x_α to the origin in the time $t' - t_\alpha$. The extremal element of problem (3.4) for which equality applies in (3.4) is the solution $s^\circ(t)$ of problem (3.3). The time $t' - t_\alpha$ in this problem is the optimal time. However, this contradicts the condition $\Lambda^\circ(t) \not\equiv \text{const}$, $t' - \varepsilon \leqslant t \leqslant t' + \varepsilon$. Hence, $s_\beta^\circ \neq 0$. Similarly, it turns out that $s_\alpha^\circ \neq 0$. Let us prove the lemma directly. Let $h_j^\circ(t) \equiv 0$ in the interval $\tau_1 \leqslant t \leqslant \tau_2$, but let $x^\circ \in X(t) - X'(t)$ for $\tau_1 \leqslant \tau' < t < \tau'' \leqslant \tau_2$. Then $\Lambda^\circ \equiv \text{const}$ for $\tau' < t < \tau''$. Let us suppose first that $\Lambda^\circ(t) \equiv \text{const}$ for $\tau' \leqslant t \leqslant t_\beta^\circ$. Then

$$h_j^\circ(t) = s_\beta^\circ S(t, t_\beta)\, b^{(j)} \equiv 0, \qquad \tau' \leqslant t \leqslant t_\beta^\circ$$

which contradicts the condition $s_\beta^\circ \neq 0$ and the property of reinforced controllability. Setting $\Lambda^\circ(t) \equiv \text{const}$ for $t_\alpha \leqslant t \leqslant \tau''$, we obtain

$$h_j^\circ(t) = s_\alpha^\circ S(t, t_\alpha) b^{(j)} \equiv 0, \qquad t_\alpha \leqslant \tau_1 < t \leqslant \tau''$$

which contradicts the controllability conditions and the property $s_\alpha^\circ \neq 0$.

Finally, let $\Lambda^\circ(t) \equiv \text{const}$ for $\tau' \leqslant t < \tau''$ and $\Lambda^\circ(t) \not\equiv \text{const}$ for $t < \tau'$, $\geqslant \tau''$. This means that

$$d^k h_j^\circ(t) = / \, dt^k = \gamma S(t, 0)\, A^k b^{(j)} \equiv 0 \qquad (k = 1, \ldots, n-1) \tag{3.5}$$

Here

$$\gamma = s_\beta^\circ S(0, t_\beta^\circ) + \int_{\tau''}^{t_\beta^\circ} d\Lambda^\circ(\xi)\, S(0, \xi)\, d\xi$$

where $\gamma \neq 0$. In fact, setting $\gamma = 0$, we obtain instead of (3.3) the problem

$$\Phi_\alpha(\Lambda^\circ{}_\alpha) + \Phi_\beta(s_\beta^\circ, \Lambda_\beta^\circ) = \min\left[\Phi_\alpha(\Lambda_\alpha) + \Phi_\beta(s_\beta, \Lambda_\beta)\right] = 1, \quad \{s_\beta, \Lambda_\alpha, \Lambda_\beta\} \tag{3.6}$$

(the minimum is taken over all the variables appearing in braces) for

$$\varphi_\alpha(\Lambda_\alpha) + \varphi_\beta(s_\beta, \Lambda_\beta) = 1 \tag{3.7}$$

Here

$\Lambda_\alpha(\xi) = \text{const}$, if $\xi \geqslant t_1$

$\Lambda_\beta(\xi) \equiv \text{const}$, if $\xi < t_2$

$$\Lambda_\beta(\xi) \not\equiv \text{const}, \quad \text{if} \quad t_2 \leqslant \xi < t_2 + \varepsilon \quad \text{for } \varepsilon > 0$$

$$\Phi_\alpha(\Lambda_\alpha) = \int_{t_\alpha}^{t_1} \gamma\left[\int_t^{t_1} d\Lambda_\alpha(\xi) S(t, \xi)\right] dt + \int_{t_\alpha}^{t_1} \gamma_1 [d\Lambda_\alpha]$$

$$\Phi_\beta(s_\beta, \Lambda_\beta) = \int_{t_2}^{t_\beta^\circ} \gamma\left[s_\beta S(t, t_\beta) + \int_t^{t_\beta^\circ} d\Lambda_\beta(\xi) S(t, \xi)\right] dt + \int_{t_\alpha}^{t_\beta^\circ} \gamma_1 [d\Lambda_\beta]$$

t_1, t_2 $(t_1 \leqslant \tau', t_2 \geqslant \tau'')$ are the first instants of emergence of $x^\circ(t)$ onto the restrictions with motion to the left from τ' or to the right from τ'',

$$\varphi_\alpha(\Lambda_\alpha) = \int_{t_\alpha}^{t_1} d\Lambda_\alpha(\xi) w(\xi) - \int_{t_\alpha}^{t_1} d\Lambda_\beta(\xi) S(t_\alpha, \xi) x_\alpha$$

$$\varphi_\beta(s_\beta, \Lambda_\beta) = s_\beta x_\beta + \int_{t_2}^{t_\beta^\circ} d\Lambda_\beta(\xi) w(\xi)$$

Problem (3.6),(3.7) is equivalent to the two problems

$$\min \Phi_\alpha(\Lambda_a) = 1 \qquad \text{for } \varphi_\alpha(\Lambda_a) = 1, \quad \{\Lambda_a\} \tag{3.8}$$

$$\min \Phi_\beta(s_\beta, \Lambda) = 1 \qquad \text{for } \varphi_\beta(s_\beta, \Lambda_\beta) = 1, \quad \{s_\beta, \Lambda_b\} \tag{3.9}$$

where

$$\Lambda_a{}^\circ = k_1 \Lambda_\alpha{}^\circ, \quad s_\beta{}^\circ = k_2 s_\beta{}^\circ, \quad \Lambda_b{}^\circ = k_2 \Lambda_\beta{}^\circ, \quad k_1 > 0, \ k_2 > 0$$

We note that (3.9) yields the solvability condition for the problem of transfer from zero to x_β in the time $t_\beta{}^\circ - t_2$. The time $t_\beta{}^\circ - t_2$ is optimal for a given problem. This contradicts the properties of the function $\Lambda_\beta{}^\circ(\xi)$. Thus, $\gamma \neq 0$, which means that condition (3.5) cannot be fulfilled. The latter contradicts out initial assumption that $h_j{}^\circ(t) \equiv 0$ for $\tau_1 \leqslant t < \tau_2$. The lemma has been proved.

Corollary 3.1. If system (1.1) is completely controllable in the reinforced sense and if at least one of the points x_α, x_β does not lie on the boundary X' of the set X, then each of the functions $h_j{}^\circ(t)$ differs from identical zero in the interval $t_\alpha \leqslant t \leqslant t_\beta{}^\circ$.

Note 3.1. The lemma is valid for all restrictions $x(t) \in X(t)$ on the coordinate convex in x.

Note 3.2. In the nonstationary variant of the lemma we must also require that the property of reinforced controllability be uniform in t.

Note 3.3. The condition $h^\circ(t) \not\equiv 0$, $t_\alpha \leqslant t \leqslant t_\beta$ is essential for justifying the convergence of the discretized variant of Problem 1.1 to the continuous variant [5]. The justification of the property adduced for regular problems in [5] is also valid in the general case.

Let us discuss a certain property of the functions $\Lambda^\circ(t)$. Let us assume that for some $j = 1, \ldots, s$ we have $h_j(t) \equiv 0$ for $\tau_1 \leqslant t < \tau_2$.

In other words, let

$$s^\circ(t) b^{(j)} \equiv 0, \qquad \tau_1 \leqslant t < \tau_2 \tag{3.10}$$

where $s^\circ(t)$ is the solution of the equation

$$s^\cdot = -sA + \lambda^\circ \tag{3.11}$$

In accordance with [6] and with Note 2.1 we interpret this equation as the equality of the distributions generated by the generalized derivatives $s^\cdot$, $\Lambda^\cdot$ of the functions s, Λ restricted by Eq. (2.12). Property (3.10) can then be understood as the equality to zero of the linear bounded operation generated by the distribution $s^\circ b^{(j)}$ when this operation is performed on infinitely differentiable functions φ which vanish outside (τ_1, τ_2): $(s^\circ b^{(j)}, \varphi) = 0$. This enables us to differentiate Eq. (3.10) (in the generalized sense). We obtain

$$\frac{d^i}{dt^i}(s^\circ(t)\, b^{(j)}) = s^\circ(t)\, A^i b^{(j)}(-1)^i + \sum_{k=1}^{i} \frac{d^{k-1}\lambda}{dt^{k-1}} A^{i-k} b^{(j)}(-1)^k \equiv 0 \quad (3.12)$$

Let

$$-A^{n+p-1} b^{(j)} = \sum_{q=0}^{n-1} \alpha_{jq}{}^{(p)} A^q b^{(j)} \quad (p = 1, \ldots, m) \qquad (3.13)$$

With allowance for (3.12) $(i = 1, \ldots, n+m-1)$, we can reduce system (3.12) to equations in the m-vector distribution λ,

$$\sum_{k=1}^{n+p-1} \frac{d^{k-1}\lambda}{dt^{k-1}} A^{n+p-k-1} b^{(j)}(-1)^k + \sum_{q=0}^{n-1} \alpha_{jq}{}^{(p)} \sum_{k=1}^{q} \frac{d^{k-1}\lambda}{dt^{k-1}} A^{q-k} b^{(j)}(-1)^k = 0 \quad (3.14)$$

which yield the necessary and sufficient condition for the fulfilment of (3.10), (3.11). By setting $\lambda = P_j \xi$ with a nonsingular matrix P_j we can reduce system (3.14) to

$$\frac{d^{n+p-2}}{dt^{n+p-2}}\xi + D^{(j)}(\xi) = 0 \qquad (3.15)$$

Here $D^{(j)}(\xi)$ is a stationary linear differential operator of order $n+p-3$.

We know, however [6], that the set of solutions of (3.15) in the class of distributions coincides with the class of solutions of (3.15) in the class of ordinary functions. This means that the quantities $\lambda^\circ(t)$ satisfy (3.10), (3.11) if and only if they are ordinary functions which constitute the solution of system of ordinary differential equations (3.14). If $h_j{}^\circ(t) \equiv 0$ for several values of j, then (3.14) must be written out for each such value of j.

We have written out system (3.14), (3.15) in general form, which means that some of the equations may be dependent. This allows us to simplify Eqs. (3.15) (e.g. in the case where not all of the $x_i{}^\circ(t)$, $i = 1, \ldots, m$ reach the restrictions or when every set of n-vectors of the form $A^i b^{(j)}$, $i = 1, \ldots, n$; $j = 1, \ldots, s$ is linearly independent). Reduction of system (3.15) to notationally simpler form for each of the possible combinations of m and r is beyond the scope of the present paper.

The simplest form of the differential equation for λ° results when $h_j{}^\circ(t) \equiv 0$ for a single j (let us say $j = 1$) and when only one of the coordinates (let us say $x_1 = f_1$) emerges onto the restriction. Then instead of (3.14) we have

$$\sum_{k=1}^{n} \frac{d^{k-1}\lambda}{dt^{k-1}} A^{n-k} b^{(1)}(-1)^k + \sum_{q=0}^{n-1} \alpha_q \sum_{k=1}^{q} \frac{d^{k-1}\lambda}{dt^{k-1}} A^{q-k} b^{(1)}(-1)^k \equiv 0 \quad (3.16)$$

$$-A^n b^{(1)} = \sum_{q=0}^{n-1} \alpha_q A^q b^{(1)}$$

Equation (3.16) is of order $n-1$ if $b_1{}^{(1)} \neq 0$. Otherwise, the order of the equation is lower than $n-1$. Summarizing the above statements, we arrive at the following statement for systems controllable in the reinforced sense.

L e m m a 3.5. If at least one of the functions $h_j{}^\circ(t)$ vanishes identically in the interval $\tau_1 \leqslant t < \tau_2$ and if system (1.1) satisfies Hypothesis 3.1, then the functions $\Lambda^\circ(t)$ are differentiable in this interval and the vector function $\lambda^\circ(t) = d\Lambda^\circ / dt$

satisfies differential equation (3.14).

Corollary 3.2. If the trajectory $x^\circ(t)$ in Hypothesis 3.1 emerges onto the restriction ($x^\circ \in X$) in a finite number of intervals e_l ($l = 1, \ldots, N$), then $\Lambda^\circ(t) = \Lambda_d^\circ(t) + \Lambda_c^\circ$, where $\Lambda_d^\circ(t)$ is piecewise-constant. The only points at which the latter can possibly experience jumps are τ_l, τ_{l+1} (the points of emergence onto and departure from the restrictions and the end points of the segments e_l). The function $\Lambda_c(t)$ is differentiable almost everywhere on $[t_\alpha, t_\beta^\circ]$.

Note 3.4. Since the functions $\lambda^\circ(t)$, $s^\circ(t)$ are continuous from the right, we can write out expressions which (by virtue of Eqs. (2.19), (3.14)) relate the initial values for the lth equation of (3.14) to the initial values for pth equations of (3.14) ($p > l$) and to the quantity s_β°,

$$\frac{d^{k_l} s(\tau_l)}{dt^{k_l}} = \zeta_l^{(k)}\left(s_\beta^\circ, \frac{ds^{k_p}(\tau_p)}{dt^{k_p}}, \quad p > l\right) \tag{3.17}$$

$$k_p = 1, \ldots, n_p, \quad l = 1, \ldots, N$$

Here $n_p + 1$ is the order of the jth equation of (3.14). Equations (3.17) are linear in s_β°, $ds^{k_p}(\tau_p) / dt^{k_p}$. If system (3.17) is nondegenerate, then the boundary conditions for each of the N equations (3.14) can be expressed in terms of s_p°, and problem (2.13), (2.14) reduced to the minimization of a function of n variables. Otherwise problem (2.13), (2.14) contains not only the unknown vector s_β, but also an additional finite number of unknown free parameters over which minimization must be carried out. Numerical realization of the above procedure for solving (2.13), (2.14) is facilitated by an upper estimate of the number N.

Note 3.5. The derivation of Corollary 3.2 remains valid if $x^\circ(t)$ emerges onto the restriction in a countable number of segments e_l provided that the set of limit points of the ends τ_l, τ_{l+1} of the segments e_l is of zero measure.

Let us formulate the necessary condition for a jump in the function $\Lambda^\circ(t)$.

Lemma 3.6. If the function $\Lambda^\circ(t)$ has a jump $\Lambda^\circ(t_1 + 0) - \Lambda^\circ(t_1 - 0) = \lambda^{(1)}$ at $t = t_1$, then

$$\lambda^{(1)} B\,(u^\circ(t_1 + 0) - u^\circ(t_1 - 0)) \geqslant 0 \tag{3.18}$$

In fact, (2.12) implies that

$$s^\circ(t_1 + 0) - s^\circ(t_1 - 0) = \lambda^{(1)} \tag{3.19}$$

On the other hand, maximum principle (2.16) implies that

$$s^\circ(t_1 + 0)\,Bu\,(t + 0) \geqslant s^\circ(t_1 + 0)\,Bu\,(t - 0) \tag{3.20}$$

$$s^\circ(t_1 - 0)\,Bu\,(t + 0) \leqslant s^\circ(t_1 - 0)\,Bu\,(t - 0) \tag{3.21}$$

Subtracting (3.21) from (3.20) and recalling (3.19), we obtain condition (3.18).

If the functions $f_i(t)$ are differentiable, then

$$\lambda^{(1)} B\,(u^\circ(t_1 + 0) - u^\circ(t_1 - 0)) \leqslant 0$$

and (3.18) can hold only if [8, 9] $x^\circ(t_1 + 0) = x^\circ(t_1 - 0)$; this is the condition of contact of the trajectory $x^\circ(t)$ and the manifold $x_i = f_i(t)$ at the point t_1. In the general case the left side of inequality (3.18) can also be strictly positive (see Example 4.4 below). Condition (3.18) is expressed geometrically by the inequality

$$\lambda^{(1)}(x^{\circ\cdot}(t + 0) - x^{\circ\cdot}(t - 0)) \geqslant 0 \tag{3.22}$$

with allowance for the fact that the vector $\lambda^{(1)}$ is supporting to $X(t_1)$ (see Note 2.2). If the sets X and U are known, then condition (3.22) allows us to isolate points capable of being associated with jumps in $\Lambda^\circ(t)$. In particular, not all the points τ_l, τ_{l+1} in the conditions of Corollary 3.2 can satisfy (3.22).

The following statement summarizes the present section.

Theorem 3.1. Let system (1.1) be completely controllable in the reinforced sense and let Hypothesis 3.1 be fulfilled. Then at least one of the functions $h_j^\circ(t) \equiv 0$ in the interval $\tau' \leqslant t < \tau''$ if and only if at least one of the identities $|x_i| \equiv f_i(t)$, $\tau' \leqslant t \leqslant \tau''$ is fulfilled. If $x^\circ(t)$ emerges onto the restriction on a finite set of segments e_l: $\tau_l \leqslant t \leqslant \tau_{l+1}$, $l = 1, \ldots, N$ only, then associated system (2.12) is of the form

$$s^{\cdot}(t) = -sA + \lambda^\circ(t) + \sum_{j=1}^{N} (\lambda^{(l)}\delta(t - t_l) + \lambda^{(l+1)}\delta(t - t_{l+1})) \qquad (3.23)$$

The function $\lambda^\circ(t)$ satisfies differential (3.14) in each interval $\tau_l \leqslant t < \tau_{l+1}$; the vectors $\lambda^{(l)}$, $\lambda^{(l+1)}$ satisfy condition (3.18) in the same intervals. The solution of problem (2.13), (2.14), which reduces here to the minimization of a function of a finite number of variables, yields the boundary conditions s_β° for system (3.23), the quantities $\lambda^\circ(t)$, $\lambda^{(l)}$, $\lambda^{(l+1)}$, and the instants τ_l, τ_{l+1} of emergence onto and departure from the restriction.

In the intervals where $h_j^\circ(t) \equiv 0$ the optimal controls can be obtained from (2.16),

$$u_j^\circ(t) = \nu_j \operatorname{sign} s^\circ(t)\, b^{(j)}$$

The form of $h^\circ(t)$ (see [10]) enables us to conclude that under the conditions of Hypothesis 3.1 each of the "relay" functions $u_j^\circ(t)$ can experience not more than $n-1$ switchings in each open interval (t_1, t_2) where $h^\circ(t) \not\equiv 0$ (and, consequently, where x does not lie on X') . But if $h_j^\circ(t) \equiv 0$, then (2.16) no longer provides information for determining $u^\circ(t)$. Additional considerations must be applied. Specifically, having solved (2.13), (2.14) and determined τ_l, τ_{l+1}, we can set $x_i = f_i(t)$ (alternatively, if the $f_i(t)$ are differentiable a sufficient number of times, we can make use of the conditions $d^k x_i / dt^k = f_j^{(k)}(t)$) either to find $u^\circ(t)$ directly or to reduce Problem 1.1 to much simpler form. We can also determine $u^\circ(t)$ by means of the device described in [5].

In conclusion we note that the above approach is also valid under nonconvex restrictions on $u(t)$. This entails the possibility of a slipping state [11, 12], and the analysis must be complemented by an appropriate interpretation of the slipping state as, for example, in [11, 13].

4. Example 4.1. Let us consider the controlled straight-line motion with friction described by the equations

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_2 + u, \quad t_\alpha = 0 \qquad (4.1)$$
$$|x_2| \leqslant 1, \quad |u| \leqslant 2, \quad x_\alpha = (0, 0) \quad x_\beta = (2 \ln {}^4/_3 + {}^1/_2,\ 0)$$

The quantity t_β° is unknown. For $|x_\beta| < 2$ the system does not have points at which $\Lambda^\circ(t)$ might experience jumps. This enables us to construct immediately problem (2.13), (2.14)

$$\min_{s_\beta, \lambda} \left\{ 2 \int_{t_\alpha}^{t_\beta} \left| s_{\beta 1}(1 - e^{-(t_\beta - \vartheta)}) + s_{\beta 2} e^{-(t_\beta - \vartheta)} + \int_{t_\alpha}^{t_\beta} \lambda(\xi) e^{-(\xi - \vartheta)} d\xi \right| d\vartheta + \int_{t_\alpha}^{t_\beta} |\lambda| \, dt \right\} = 1 \qquad (4.2)$$

for $s_{\beta_1} x_{\beta_1} = c$ (for convenience we set $c = x_{\beta_1}$). Recalling the fact that Eq. (3.16) is here of the form $\lambda^{\cdot} = 0$, $\lambda = \mathrm{const}$ and constructing the equation of the type (3.17)

$$s_{\beta 1}(1-e^{-(t_\beta-\tau_1)})+s_{\beta 2}e^{-(t_\beta-\tau_1)}+\lambda\int_{\tau_1}^{\tau_2}e^{-(\xi-\vartheta)}\,d\xi=0 \qquad (4.3)$$

expressing λ in terms of $s_{\beta 1}$, $s_{\beta 2}$, we obtain the solution of problem (4.2) and then

$$\tau_1=\ln 2,\quad \tau_2=1+\ln 2,\quad t_\beta^\circ=1+\ln 3,\quad s_\beta^\circ=(2,-1)$$
$$\lambda^\circ(t)=-2,\quad \tau_1\leqslant t<\tau_2,\quad \lambda^\circ\equiv 0,\ t\leqslant\tau_1,\ t>\tau_2$$

The optimal equation satisfying the maximum principle is in this case

$$u^\circ(t)=2,\quad 0\leqslant t<\tau_1,\qquad u^\circ(t)=x_2^\circ(t)=1,\quad \tau_1\leqslant t<\tau_2$$
$$u^\circ(t)=-2,\quad \tau_2\leqslant t\leqslant t_\beta^\circ$$

Example 4.2. Let us consider the oscillatory motion with a resistance force described by the equations $x_1^{\cdot}=x_2,\quad x_2^{\cdot}=-5x_1-2x_2+2u,\quad t_\alpha=0$ (4.4)

$$|x_2|\leqslant 1,\quad |u|\leqslant 5,\quad x_\alpha=(0,0),\quad x_\beta=(1.2;0)$$

Here jumps in $\Lambda^\circ(t)$ can occur at the points $(\pm 1.6;\pm 1)$, $(\pm 2.4;\mp 1)$. Under the above boundary conditions we can look for the quantity λ° in the class of piecewise-absolutely-continuous functions. We have the problem

$$\min\Big\{5\int_{t_\alpha}^{t_\beta^\circ}\Big|e^{-(t_\beta^\circ-\vartheta)}\Big[\frac{s_{\beta 1}}{2}\sin 2(t_\beta^\circ-\vartheta)+s_{\beta 2}(\cos 2(t_\beta^\circ-\vartheta)-$$
$$-\frac{1}{2}\sin 2(t_\beta^\circ-\vartheta)\Big]+\int_{\vartheta}^{t_\beta^\circ}\lambda(\xi)\Big[\cos 2(\xi-\vartheta)-\frac{1}{2}\sin 2(\xi-\vartheta)\Big]d\xi\Big|\,d\vartheta+$$
$$+\int_{t_\alpha}^{t_\beta^\circ}|\lambda|\,d\xi\Big\}=1,\quad \{s_\beta,\lambda\}\quad \text{for}\quad s_{\beta_1}x_{\beta_1}=c=x_{\beta 1} \qquad (4.5)$$

Equation (3.16) is of the form $\dot\lambda=4\lambda$, which means that

$$\lambda^\circ(t)=k^\circ e^{4(t-\tau_1)}\quad \text{for}\ \tau_1\leqslant t<\tau_2,\ \lambda^\circ(t)\equiv 0\quad \text{for}\ t<\tau_1,\ t\geqslant\tau_2$$

Constructing the equation analogous to (4.3) and solving (4.5), we obtain

$$s_{\beta_1}^\circ=1,\quad s_{\beta_2}^\circ=\sin 2(t_\beta^\circ-\tau_2)\,/\,5\sin[2(t_\beta^\circ-\tau_2)-\alpha]$$
$$\cos\alpha={}^1/_5,\quad \sin\alpha={}^2/_5,\quad k^\circ={}^2/_3e^{-(t_\beta^\circ-\tau_2)}\csc(2t_\beta^\circ-2\tau_2-\alpha),$$
$$\tau_1=0.113,\quad \tau_2=1.163,\quad t_\beta^\circ=1.222,\quad u^\circ(t)=5,\quad 0\leqslant t<\tau_1$$
$$u^\circ(t)=2.5x_1^\circ+x_2^\circ,\quad \tau_1\leqslant t<\tau_2,\quad u^\circ(t)=-5,\quad \tau_2\leqslant t\leqslant t_\beta^\circ$$

We can see that problems 4.1 and 4.2 are regular. Let us consider some sample irregular problems.

Example 4.3. We have the equations

$$x_1^{\cdot}=x_2,\qquad x_2^{\cdot}=u \qquad (4.6)$$

$$|u|\leqslant 1,\qquad |x_2|\leqslant 1+{}^1/_2(4.5-t)^2=f(t),\quad x_\alpha=(0,0),\qquad x_\beta=({}^{163}/_{24},\ 0)$$

Verifying the possibility of fulfilment of condition (3.18) at some point, we see that $t_1=3.5$, $t_2=5.5$ are such points. This means that if t_1, t_2 are points of emergence onto or departure from the restriction, then we can expect jumps in $\Lambda^\circ(t)$ at these points. We therefore attempt immediately to find $\lambda^\circ(t)$ in the form

$$\lambda^\circ(t)=\lambda^{(1)}\delta(t-t_1)+\lambda^{(2)}\delta(t-t_2)+\lambda,\ \tau_1\leqslant t\leqslant\tau_2\ (\lambda=\text{const})$$

since (3.16) is of the form $\lambda^{\cdot}(t) = 0$

$$\min\left\{\int_0^{t_\beta^\circ}\left|s_{\beta 1}(t_\beta^\circ - \vartheta) + s_{\beta 2} + \int_\vartheta^{t_\beta^\circ}\lambda(t)\,dt\right|d\vartheta + \int_0^{t_\beta^\circ} f(t)\,|\lambda(t)|\,dt\right\} = 1\,\{s_\beta, \lambda\} \qquad (4.7)$$

for $s_{\beta 1}x_{\beta 1} = x_{\beta 1}$. Solving the above problem, we obtain

$$\tau_1 = 3.5,\ \tau_2 = 4.5,\ t_\beta^\circ = 5.5,\ \lambda^{(1)} = -1,\ \lambda^{(2)} = 0,\ \lambda = -s_{\beta 1}^\circ,\ s_\beta^\circ = (1, -1)$$

The function $h^\circ(\vartheta) = s^\circ(\vartheta)\, b$ is therefore discontinuous,

$$h^\circ = 2.5 - \vartheta \quad (0 \leqslant \vartheta < 3.5), \qquad h^\circ \equiv 0 \quad (3.5 \leqslant \vartheta < 4.5)$$
$$h^\circ(\vartheta) = 4.5 - \vartheta \quad (4.5 \leqslant \vartheta \leqslant t_\beta^\circ = 5.5)$$

The control $u^\circ(t)$ is of the form

$$u^\circ(\vartheta) = 1 \ (0 \leqslant \vartheta < 2.5), \quad u^\circ = -1 \ (2.5 \leqslant \vartheta < 3.5), \quad u^\circ = 0 \ (3.5 \leqslant \vartheta < 4.5)$$
$$u^\circ = -1\ [4.5, 5.5]$$

We note that by virtue of the smoothness of $f(t)$ the optimal trajectory $x^\circ(t)$ comes in contact with the restriction X at $t = 3.5$.

Example 4.4. Let us suppose that the continuous restriction in Example 4.3 is not smooth: $|x_2| \leqslant at + b$, where $a = 0, b = 1$ as long as $0 \leqslant t \leqslant 2$, and, further, $a > 1$, $b = 1 - 2a$ if $t > 2$. $x_\alpha = (0,0)$ $x_\beta = (5,0)$, $|u| \leqslant 1$. The only point where $\Lambda^\circ(t)$ can experience a jump is $t_1 = 2$. Solving problem (4.7) for $f(t) = at + b$, we obtain $\lambda^\circ = \lambda^{(1)}\delta(t - t_1) + \lambda$ ($\lambda = \text{const}$), $\tau_1 \leqslant t \leqslant \tau_2$. As a result we obtain $s_\beta^\circ = (1, -2)$, $\lambda^{(1)} = -1$, $\lambda = -s_\beta^\circ$, $\tau_1 = 1$, $\tau_2 = 2 = t_2$ and, moreover, $t_\beta^\circ = 5$.

$$s^\circ(t)\,b = 1 - t, \quad u^\circ(t) = 1 \quad (0 \leqslant t < 1)$$
$$s^\circ(t)\,b = 0, \quad u^\circ(t) = 0 \quad (1 \leqslant t < 2)$$
$$s^\circ(t)\,b = 3 - t, \quad u^\circ(t) = \operatorname{sign} s^\circ(t)\,b \quad (2 \leqslant t \leqslant 5)$$

i.e.

$$u^\circ = 1 \quad (2 \leqslant t < 3), \quad u^\circ = -1 \ (3 \leqslant t \leqslant 5)$$

In this case $x^\circ(t)$ does not come in contact with the restriction X at $t_2 = \tau_2$. Like $f(t)$, the trajectory $x^\circ(t)$ is not differentiable for $t = 2$.

Example 4.5. Now let us suppose that Hypothesis 3.1 is not fulfilled; for the system $x_1^{\cdot} = x_2$, $x_2^{\cdot} = u$, $|u| \leqslant 1$ we are given the condition $|x_2| \leqslant f(t)$, where $f(t) = 5 - t$ for $0 \leqslant t \leqslant 3$ and $f(t) = 2 + t$ for $t > 3$, $x_\alpha = (0, 0)$, $x_\beta = (7,0)$. This problem is similar to (4.7). However, we cannot make use of Eq. (3.16) in this case, since we do not necessarily have $h^\circ(t) \equiv 0$ at the restriction. The quantity (generalized function) $\lambda^\circ(t)$ can be obtained directly by minimizing the expression of the form (4.7). This yields

$$s_\beta^\circ = (1, -2), \quad \lambda^\circ(t) = -2\delta(t - 3), \quad t_\beta^\circ = 6$$
$$h^\circ(t) = 2 - t \quad (0 \leqslant t < 3), \qquad h^\circ(t) = 4 - t \quad (3 \leqslant t \leqslant 6)$$
$$u^\circ(t) = 1, \quad (0 \leqslant t < 2,\ 3 \leqslant t < 4)$$
$$u^\circ(t) = -1, \quad (2 \leqslant t < 3,\ 4 \leqslant t < 6)$$

Example 4.6. Let us again consider Eqs. (4.6), but under the conditions $|u| \leqslant 1$, $|x_2| \leqslant f(t)$, where the function $f(t)$ is defined as follows. We are given the function

$$\varphi(t) = \begin{cases} 3 - t - 2^{1-n}, & (1 - 2^{1-n} \leqslant t \leqslant 1 - 2^{-n} - 2^{-n-1}) \\ t - 1 + 2^{1-n}, & (1 - 2^{1-n} - 2^{-n-1} \leqslant t \leqslant 1 - 2^{-n}) \\ 1, & 0 \leqslant t \leqslant 1, \quad 2 \leqslant t \leqslant 3 \end{cases}$$
$$(n = 1, \ldots, N, \ldots)$$

Moreover, the function $f(t)$ is defined in such a way that $f(t) > \varphi(t)$ for $1-2^{1-n}-2^{-n-2} < t < 1-2^{1-n}+2^{-n-2}$ and $f(t) = \varphi(t)$ at the remaining points. The boundary conditions are as follows: $x_\alpha = (0,0)$, $x_\beta = (0, {}^{33}/_{12})$. Again minimizing an expression of the form (4.7) under the condition $s_{\beta_1} = 1$, we obtain

$$t_\beta{}^\circ = 3, \quad s_{\beta 1}{}^\circ = 1, \quad s_{\beta 2}{}^\circ = -1, \quad \lambda^\circ(t) = \sum_{i=1}^{\infty} \alpha_i \delta(t - t_i)$$

$$\alpha_i = 2^{-i}, \ t_i = 1 - 2^{-i} - 2^{-i-1}$$

Computing the quantity $h^\circ(t)$ (which differs from identical zero everywhere in this case), we obtain

$$u^\circ(t) = 1, \qquad (0 \leqslant t \leqslant 1,\ 1-2^{-(n-1)}-2^{-(n+1)} \leqslant t < 1-2^{-n})$$

$$u^\circ(t) = -1, \qquad (2 \leqslant t \leqslant 3,\ 1-2^{-(n-1)} \leqslant t \leqslant 1-2^{-n}-2^{-(n-1)})$$

In this example the motion $x^\circ(t)$ emerges onto the restriction an infinite number of times and the function $\Lambda^\circ(t)$ has a countable number of jumps. The optimal control can be determined everywhere on the basis of the maximum principle. One of the reasons for this is the fact that Hypothesis 3.1 is not fulfilled.

In conclusion we note that the solution of problems with an infinite number of emergences onto the restriction is not usually reducible to the minimization of functions of a finite number of variables, although the procedure described in Sect. 2 remains valid for such problems. The chief difficulty in this case lies in the determination of the minimizing element from (2.13), (2.14).

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